

# Mathematics 222B Lecture 23 Notes

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## 1 Nöether's Principle and the Energy-Momentum Tensor

### 1.1 Nöether's principle

Let's continue our discussion of Nöether's principle with an updated version of the slogan we gave last time. The slogan for the principle is '(continuous) symmetries give rise to conservation laws.' The implication in the other direction is not always the case; for more on the reverse, you can see, for example, Carter's constant, which is a "hidden symmetry" for geodesics on Kerr spacetime.

**Theorem 1.1.** *Consider the Lagrangian action  $F[u] = \int_U L(Du, u, x) dx$ . Suppose there exists a continuous symmetry  $(u_\tau(x), X_\tau(x))$  of the action (with  $u_\tau : U \rightarrow \mathbb{R}$  and  $X_\tau : \mathbb{R}^d \rightarrow \mathbb{R}^d$  a diffeomorphism for each  $\tau$ ), in the sense that*

$$\int_U L(Du_\tau(x), u_\tau(x), x) dx = \int_{U(\tau)} L(Du, u, x) dx,$$

where  $U(\tau) := X_\tau(U)$ . Then

$$\partial_{x^j}(m \partial_{p_j} L(Du, u, x) - L(Du, u, x) V^j) = m \left( \frac{\partial}{\partial x^j} (\partial_{p_j} L(Du, u, x)) - \partial_z L(Du, u, x) \right),$$

where  $m = \frac{\partial}{\partial \tau} u|_{\tau=0}$ ,  $u = u_\tau|_{\tau=0}$ ,  $V^j = \frac{\partial}{\partial \tau} X_\tau^j|_{\tau=0}$ , and  $X_0(x) = x$ .

**Lemma 1.1.** *Let  $f_\tau = f_\tau(x)$ , and let  $U_\tau$  be a "smooth" family of  $C^\infty$  domains, i.e. there exist a family of diffeomorphisms  $X_\tau : \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that  $U_\tau = X_\tau(U)$ . Let  $V(x) = \frac{\partial}{\partial \tau} X_\tau(x) \Big|_{\tau=0}$  for  $x \in \partial U_0$ . Then*

$$\frac{d}{d\tau} \int_{U_\tau} f_\tau(x) dx \Big|_{\tau=0} = \int_{U_0} \frac{\partial}{\partial \tau} f_\tau(x) \Big|_{\tau=0} dx + \int_{\partial U_0} f_0 V \cdot \nu.$$

Here is the proof of the theorem, assuming the lemma:

*Proof.*

$$\begin{aligned} \left. \frac{\partial}{\partial \tau} (\text{LHS}) \right|_{\tau=0} &= \frac{\partial}{\partial \tau} \int_U L(Du_\tau(x), u_\tau(x), x) dx \Big|_{\tau=0} \\ &= \int_U \frac{\partial}{\partial \tau} L(Du_\tau(x), u_\tau(x), x) \Big|_{\tau=0} dx \end{aligned}$$

Using the Euler-Lagrange equation,

$$= \int_U \frac{\partial}{\partial_j} L \cdot \partial_{x^j} m + \frac{\partial}{\partial z} L \cdot m dx$$

Integrating by parts,

$$= \int_U \left( -\partial_{x^j} \left( \frac{\partial}{\partial p_j} L \right) + \frac{\partial}{\partial z} L \right) m dx + \int_{\partial U} \frac{\partial}{\partial p_j} L \cdot m \nu_j dA.$$

The lemma gives

$$\left. \frac{\partial}{\partial \tau} (\text{RHS}) \right|_{\tau=0} = \int_{\partial U} LV^j \nu_j dA.$$

Putting these together, we get

$$\int_{\partial U} \left( \frac{\partial}{\partial p_j} L \cdot m - LV^j \right) \nu_j dA = \int_U \left( -\frac{\partial}{\partial x^j} (\partial_{p_j} L) + \partial_z L \right) m dx.$$

By the divergence theorem, the left hand side is

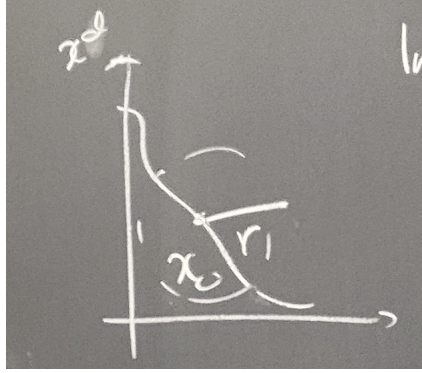
$$\int_U \partial_{x^j} \left( \frac{\partial}{\partial p_j} L \cdot m - LV^j \right) dx.$$

since  $U$  is arbitrary. □

Here is a proof of this lemma, using the fact that the derivative of the Heaviside function is the delta distribution. (A more standard way to prove this is to use a change of variables to turn one of the integrals into a volume integral.)

*Proof.* Here is a sketch of the idea. Without loss of generality, let  $f_\tau = f$ , where  $f \in C_c^\infty(\mathbb{R}^d)$  and  $\text{supp } f \subseteq B_r(x_0)$ . Choose  $x_0$  so that  $U \cap B_r(x_0) = \{x^d > \gamma(x^1, \dots, x^{d-1})\}$ .

So  $X_{-\tau}^d - \gamma_\tau(X_{-\tau}')$  is the defining function for  $\partial U_\tau$ .



Then

$$\int_{U_\tau} f \, dx = \int \mathbb{1}_{U_\tau} f \, dx = \int H(x^n - \gamma_\tau(x')) f(x) \, dx.$$

Now we can differentiate

$$\begin{aligned} \frac{\partial}{\partial \tau} \int H(\underbrace{X_\tau^d - \gamma_\tau(X_\tau')}_{u(X_{-\tau})}) f(x) \, dx \Big|_{\tau=0} &= \int H'(\underbrace{X_\tau^n - \gamma_\tau(x')}_{u(X_{-\tau})}) \underbrace{\frac{\partial}{\partial \tau}(x^n - \gamma_\tau(x'))}_{\frac{\partial}{\partial \tau} u(X_{-\tau})|_{\tau=0}} \cdot f(x) \, dx \\ &= \int \delta_0(U(X_0)) \underbrace{\partial_j u \cdot \frac{\partial}{\partial \tau} x^j}_{\nabla u \cdot V^j} \Big|_{\tau=0} \cdot f(x) \, dx \end{aligned}$$

The  $\delta_0$  part gives us the surface measure on  $\partial U$  times  $\frac{1}{|\nabla u(x)|}$

$$= \int_{\partial U} f \underbrace{\frac{(-\nabla u)}{|\nabla u|}}_\nu \cdot V \, dA. \quad \square$$

**Remark 1.1.** In the view of distribution theory, the divergence theorem is precisely telling us about the derivative of this kind of indicator function.

**Example 1.1.** Consider the action

$$F[\phi] = \int -\frac{1}{2} |\partial_t \phi|^2 + \frac{1}{2} |D_x \phi|^2 \, dx,$$

so  $L = -\frac{1}{2} p_0^2 + \frac{1}{2} |p_x|^2$ . Let  $\phi : \mathbb{R}^{1+d} \rightarrow \mathbb{C}$ , and let  $\phi_\tau(x) e^{i\tau} u(x)$  and  $X_\tau(x) = x$ . Then Nöether's principle tells us that there is an associated conservation law for the wave equation:  $\partial_\mu J^\mu = 0$ , where

$$J^0 = \text{Im}(\phi \overline{\partial_t \phi}), \quad J^j = \text{Im}(\phi \overline{\partial_j \phi}).$$

This is called the conservation of the **charge-current vector**.  $J^0$  is the natural charge density, and  $J^j$  is the natural wave density if we want to couple the wave equation with Maxwell's equations.

In the case of the Schrödinger equation, this type of computation was carried out by Weyl. This gives rise to **gauge theory**. More examples can be found in Evans' book.

## 1.2 The energy-(stress)-momentum tensor

Here is useful alternate formulation of Nöether's principle. Our setting now is that  $U \subseteq \mathcal{M}$ , where  $\mathcal{M}$  is a manifold with metric  $g$  ( $g$  may be Riemannian or Lorentzian or pseudo-Riemannian). Assume that

$$L(Du, u, x) = \mathcal{L}(du, u, g) \sqrt{|\det g|},$$

so the action looks like

$$F[u] = \int_U \mathcal{L}(du, u, g) \sqrt{|\det g|} dx.$$

This is invariant under change of coordinates, and the claim is that Nöether's principle will give us a conserved quantity that we call the energy-momentum tensor.

**Theorem 1.2.** *Assume that  $F$  is of the above form, and define*

$$T^{\mu, \nu} = \frac{\partial}{\partial g_{\mu, \nu}} \mathcal{L} + \frac{1}{2} (g^{-1})^{\mu, \nu} \mathcal{L}.$$

*Then the covariant derivative associated with  $g$  satisfies*

$$\nabla_\mu T^{\mu, \nu} = 0$$

*if  $u$  satisfies the Euler-Lagrange equation.*

*Proof.* Consider a compactly supported 1-parameter family of diffeomorphisms  $X_\tau : U \rightarrow U$  such that  $X_0(x) = x$  and such that for all  $\tau$ ,  $X_\tau(x) = x$  outside some  $K \subseteq\subseteq U$ . The invariance looks like

$$\int_U \mathcal{L}(du, u, g) \sqrt{|\det g|} dx = \int_U \mathcal{L}(d(u \circ X_\tau), u \circ X_\tau, X_\tau^* g) \sqrt{|\det X_\tau^* g|} dx$$

Now

$$\left. \frac{d}{d\tau} (\text{LHS}) \right|_{\tau=0} = 0,$$

whereas

$$\left. \frac{d}{d\tau} (\text{RHS}) \right|_{\tau=0} = \underbrace{\left( \frac{\partial}{\partial \tau} \text{falls on } u \circ X_\tau \right)}_I + \underbrace{\left( \frac{\partial}{\partial \tau} \text{falls on } X_\tau^* g \right)}_{II}$$

Term  $I$  vanishes via the Euler-Lagrange equation. In fact,  $I = \int (-\partial_\mu (\frac{\partial}{\partial p_\mu} \mathcal{L}) + \partial_z \mathcal{L}) V u \, dx$ , where  $V(x) = \frac{\partial}{\partial \tau} X_\tau(x)|_{\tau=0}$ . For term  $II$ , we have

$$\begin{aligned} \int \frac{\partial}{\partial g_{\mu,\nu}} \mathcal{L}(du, u, g) \frac{\partial}{\partial \tau} X_\tau^* g_{\mu,\nu} \Big|_{\tau=0} \sqrt{|\det g|} \\ + \frac{1}{2} \mathcal{L}(du, u, g) \cdot \underbrace{\frac{\det g}{|\det g|} \frac{\frac{\partial}{\partial \tau} \det X_\tau^* g|_{\tau=0}}{|\det g|}}_{\frac{\partial}{\partial \tau} \log \det X_\tau^* g|_{\tau=0}} \sqrt{|\det g|} \, dx, \end{aligned}$$

where we have used

$$\begin{aligned} \frac{\partial}{\partial \tau} \sqrt{|\det g_\tau|} \Big|_{\tau=0} &= \frac{1}{2} \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial \tau} |\det g_\tau| \\ &= \frac{1}{2} \frac{\det g}{|\det g|} \partial_\tau \det g_\tau|_{\tau=0} = \frac{1}{2} \frac{1}{\det g} \partial_\tau (\det g_\tau)|_{\tau=0} \sqrt{|\det g|}. \end{aligned}$$

From elementary differential geometry, we have a name for this: this is the Lie derivative

$$\frac{\partial}{\partial \tau} (X_\tau^* g)_{\mu,\nu} \Big|_{\tau=0} = \mathcal{L}_V g_{\mu,\nu} = \nabla_\mu V_\nu + \nabla_\nu V_\mu.$$

How do we differentiate the determinant function? First, note that we can differentiate near the identity:

$$\frac{\partial}{\partial \tau} \det(I + \tau A) \Big|_{\tau=0} = \text{tr}(A).$$

Now if we let  $B_0 = I$  and  $\frac{\partial}{\partial \tau} B_\tau|_{\tau=0} = A$ , then

$$\frac{\partial}{\partial \tau} \det(B_\tau) \Big|_{\tau=0} = \frac{\partial}{\partial \tau} \det(I + \tau A + O(\tau^2)) \Big|_{\tau=0} = \text{tr } A.$$

Now if  $C_0 = M$  (which is invertible) and  $\frac{\partial}{\partial \tau} C_\tau \Big|_{\tau=0} = A'$ , then

$$\begin{aligned} \frac{\partial}{\partial \tau} \det(C_\tau) \Big|_{\tau=0} &= \frac{\partial}{\partial \tau} \det(M^{-1}(\tau)) \Big|_{\tau=0} \det M \\ &= \det M \, \text{tr}(M^{-1} A'), \end{aligned}$$

so that

$$\frac{\partial}{\partial \tau} \log \det C_\tau \Big|_{\tau=0} = \text{tr}(M^{-1} A').$$

Now we can deal with the term  $\frac{\partial}{\partial \tau} \log \det X_\tau^* g|_{\tau=0}$  as

$$\frac{\partial}{\partial \tau} \log \det X_\tau^* g|_{\tau=0} = \text{tr}(g^{-1} \mathcal{L}_V g) = (g^{-1})^{\mu,\nu} (\mathcal{L}_V g)_{\mu,\nu}$$

All in all, we see that

$$\begin{aligned}
II &= \int \underbrace{\left( \frac{\partial}{\partial g_{\mu,\nu}} \mathcal{L} \frac{1}{2} (g^{-1})^{\mu,\nu} \mathcal{L} \right)}_{T^{\mu,\nu} = T^{\nu,\mu}} \underbrace{\mathcal{L}_V g_\mu}_{\nabla_\mu V_\nu + \nabla_\nu V_\mu} \sqrt{|\det g|} dx \\
&= 2 \int_U T^{\mu,\nu} \nabla_\mu V_\nu \sqrt{|\det g|} dx \\
&= -2 \int_U (\nabla_\mu T^{\mu,\nu}) V_\nu \sqrt{|\det g|} dx \\
&= 0
\end{aligned}$$

for all  $X_\tau$ . Thus,  $\nabla_\mu T^{\mu,\nu} = 0$ . □

**Example 1.2** (E-M for Laplace/wave equation). Here,  $\mathcal{L} = (g^{-1})^{\mu,\nu} \partial_\mu u \partial_\nu u$ , so

$$T^{\mu,\nu} = \partial_\mu u \partial_\nu u - \frac{1}{2} g_{\mu,\nu} \partial^\nu u \partial_\nu u.$$

We can see that

$$\frac{\partial}{\partial g_{\mu,\nu}} \mathcal{L} = \frac{\partial}{\partial g^{\mu,\nu}} (g^{-1})^{\mu',\nu'} \frac{\partial}{\partial (g^{-1})^{\mu',\nu'}} \mathcal{L} = -(g^{-1})^{\mu,\mu'} (g^{-1})^{\nu,\nu'} \frac{\partial}{\partial (g^{-1})^{\mu',\nu'}} \mathcal{L}.$$